Entanglement capabilities of non–local Hamiltonians

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We quantify the capability of creating entanglement for a general physical interaction acting on two qubits. We give a procedure for optimizing the generation of entanglement. We also show that a Hamiltonian can create more entanglement if one uses auxiliary systems.

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In the last 2-3 years there has been very considerable increase in experimental activity aiming to create entangled quantum states. One reason is the potential applications of entanglement to quantum information processing. Creating entanglement has been possible in quantum optics for more than a decade, however now many new communities, working in a variety of experimental areas (for example NMR, condensed matter physics) are also joining the field [1]. In general, entanglement between two systems can be generated if they interact in a controlled way. However, in most experiments these interactions are weak which make the production of entanglement a very difficult task. Thus, it would be very convenient to have a theory which would provide us with the best way of exploiting interactions to produce entanglement.

In this Letter we analyze the entanglement capabilities of Hamiltonians. In particular, we would like to answer questions like: given an interaction (Hamiltonian), what is the most efficient way of entangling particles? Can we make the process more efficient by supplementing the action of the Hamiltonian with some local unitary operations? Can we increase the entanglement more efficiently by using some ancillas?

So far, much of the theoretical effort in quantum information theory has been devoted to the characterization and quantification of the entanglement of a given state. Very recently, it has been realised that there is a parallel notion of the entanglement in the dynamics of a system [2]. In [2], the authors consider the situation that one has a given unitary transformation and ask, for example, how much state entanglement is needed to produce it. Here we focus on a different issue: given an *interaction* (i.e. a Hamiltonian) how can we make the most effective use of it [3]. What we propose here is to define and determine the entanglement capabilities of physical processes, in particular, of unitary evolutions [4]. This is a very relevant problem not only from the theoretical point of view, but also from the experimental one. Of course, this problem is even more difficult than the one of quantifying the entanglement of states. In any case, in the present work we give the first steps in this direction by considering the case in which the physical process is acting on two qubits.

From our results it turns out that: (i) It is more efficient to produce entanglement if one initially has already some; (ii) The best initial entanglement is universal, i.e. independent of the physical process; (iii) One can improve the performance of a physical process by complementing it with fast local operations; (iv) One can also improve it (in certain cases) by using auxiliary systems; (v) All entangling Hamiltonians can simulate each other and are thus qualitatively equivalent; we also provide an upper bound on the time required for one Hamiltonian to simulate another.

We consider two qubits interacting via a non–local Hamiltonian H. We want to determine the most efficient way in which we can use such an interaction to produce entanglement. We will characterize the entanglement of a state of the qubits at a given time t, $|\Psi(t)\rangle$, by some entanglement measure E. In order to quantify the entanglement production, we define the *entanglement rate* Γ at a particular time t of the interaction as follows:

$$\Gamma(t) \equiv \frac{dE(t)}{dt}. \tag{1}$$

This quantity depends on $|\Psi(t)\rangle$ not only through its entanglement E. The goal is then to find the conditions which must be satisfied in order to obtain a maximal entanglement rate. In particular, we will be interested in determining the following:

(i) For any initial entanglement E of the two-qubit system, what is the state $|\Psi\rangle$, say $|\Psi_E\rangle$, for which the interaction produces the maximal rate Γ_E.

(ii) The maximal achievable entanglement rate $\Gamma_{\max}$ [5], $\Gamma_{\max} \equiv \max_E \Gamma_E$ and the state $|\Psi_{\max}\rangle$ for which $\Gamma = \Gamma_{\max}$.

These quantities are interesting because the knowledge of the state $|\Psi_E\rangle$ will allow us to find out the most efficient way of entangling the qubits. The idea is to supplement the interaction Hamiltonian H with appropriate local unitary operations in such a way that the state of the qubits at any time t is precisely $|\Psi_{E(t)}\rangle$, for which the

increase of entanglement is optimal. In order to show how this can be achieved, let us consider that the evolution given by H proceeds in very small time steps δt. Let us also assume that the qubits are initially disentangled. Using local operations, we can always prepare the state $|\Psi_0\rangle$ —that is, the product state which most efficiently becomes entangled under the action of H. After a time step δt, the state will change and its entanglement will increase to δE. Then, we use (fast) local unitary operations to transform the new state of the qubits into the state $|\Psi_{\delta E}\rangle$ for which Γ is optimal. Note that this is always possible, since for qubits all states with the same value of E, say δE, are connected by local unitary transformations. By proceeding in the same way after every time step, and taking the continuous time limit $\delta t \to 0$ we obtain that the state of the qubits at time t is always the optimal one, $|\Psi_{E(t)}\rangle$. Obviously, in an experimental realization, this procedure requires that we can apply the appropriate local transformations in times which are short compared to the typical time scale τ_H associated to H, $\tau_H = (e_{\max} - e_{\min})^{-1}$, where $e_{\max}$ and $e_{\min}$ are the maximum and minimum eigenvalues of H, and we have set $\hbar = 1$.

Knowledge of Γ_E also permits us to determine the maximum amount of entanglement $E_{\max}$ produced as a function of time. We just have to express Γ_E as an explicit function of E, substitute it in (1) and solve that differential equation to determine $E_{\max}(t)$. Note that the optimal procedure described above will precisely reach the entanglement $E_{\max}(t)$.

The state $|\Psi_{\max}\rangle$ is important since it gives rise to the maximal increase of entanglement, and therefore corresponds to the best operational point. After reaching the state $|\Psi_{\max}\rangle$ with the procedure described in the previous paragraph, the entanglement would be produced in a very efficient way, if one could transfer the entanglement that is gained after each time step δt to other qubits (using entanglement dilution [6] or some other means). In particular, it would increase proportionally to the time, $\Gamma_{\max}$ being the proportionality constant.

In the following, we will show how to determine $|\Psi_E\rangle$, Γ_E, $|\Psi_{\max}\rangle$, and $\Gamma_{\max}$ for an arbitrary Hamiltonian H. To this end, it is convenient to use the Schmidt decomposition of the state of the qubits $|\Psi(t)\rangle$ to write

$$|\Psi\rangle = \sqrt{P}|\varphi,\chi\rangle + e^{i\alpha}\sqrt{1-P}|\varphi^\perp,\chi^\perp\rangle, \tag{2}$$

where for the sake of short–hand notation we have omitted the time dependence of all these quantities. Here, $\langle\varphi|\varphi^\perp\rangle = \langle\chi|\chi^\perp\rangle = 0$ and $P \leq 1/2$. Note that E must only depend on the Schmidt coefficient P, given the fact that it must be invariant under local unitary operations. For example, if we choose as entanglement measure the entropy of entanglement [6] — the entropy of the reduced density operator of one of the qubits—, we will have

$$E(P) = -P\log_2(P) - (1-P)\log_2(1-P). \tag{3}$$

Note that the entropy of entanglement quantifies the amount of EPR entanglement contained asymptotically in a pure state $|\Psi\rangle$. That is $E(P)$ gives the ratio of maximally entangled EPR states $|\Psi^-\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle)$ which can be distilled from [are needed to create] $|\Psi\rangle$ respectively. Thus, we can write

$$\Gamma(t) = \frac{dE}{dP}\frac{dP}{dt}. \tag{4}$$

In (4), given a particular entanglement measure $E(P)$, we just have to determine dP/dt. In order to do that, we need to find the (infinitesimal) time evolution of the Schmidt coefficients of the state of the qubits. After a time δt we will have $|\Psi(t+\delta t)\rangle = \exp(-iH\delta t)|\Psi(t)\rangle \simeq (1 - iH\delta t)|\Psi(t)\rangle$. The corresponding reduced density operator $\rho_A(t+\delta t)$, where $\rho_{A,B} = \mathrm{Tr}_{B,A}(|\Psi\rangle\langle\Psi|)$, can then be written as $\rho_A(t+\delta t) = \rho_A(t) - i\delta t\mathrm{Tr}_B\{[H, |\Psi(t)\rangle\langle\Psi(t)|]\}$. The eigenvalues (Schmidt coefficients) of this operator can be easily determined starting from $\rho_A|\varphi\rangle = P|\varphi\rangle$ and using standard perturbation theory. We find

$$\frac{dP}{dt} = 2\sqrt{P(1-P)} \times \mathrm{Im}[e^{i\alpha}\langle\varphi,\chi|H|\varphi^\perp,\chi^\perp\rangle], \tag{5}$$

where we have omitted the time–dependence. Upon substitution in (4) we obtain the entanglement rate. Since we are interested in maximizing Γ, it is clear that we can always choose α such that

$$\Gamma = f(P)|h(H,\varphi,\chi)|. \tag{6}$$

where

$$f(P) = 2\sqrt{P(1-P)}E'(P), \tag{7a}$$

$$h(H,\varphi,\chi) = \langle\varphi,\chi|H|\varphi^\perp,\chi^\perp\rangle. \tag{7b}$$

By analyzing Eq. (6) we can extract some interesting conclusions, even before determining the maximum value of Γ explicitly. Given the fact that f and h depend on different parameters, in order to determine the quantities mentioned in (i–ii) we can maximize the functions f and $|h|$ independently. First, if we want to determine the quantities mentioned in (i), we have to fix the value of E. In that case, P is also fixed and therefore the maximum of the entanglement rate will correspond to a state of the form (2) with some fixed $|\varphi\rangle$, $|\chi\rangle$, and α (which maximize $|h|$). That is, for any value E of the entanglement, the states $|\varphi\rangle$ and $|\chi\rangle$ for which the maximal entanglement rate Γ_E is obtained do not depend on E, but only on the form of the Hamiltonian H. Let us denote by $h_{\max}$ the maximum value of $|h|$; that is,

$$h_{\max} = \max_{||\varphi||,||\chi||=1} |\langle\varphi,\chi|H|\varphi^\perp,\chi^\perp\rangle|. \tag{8}$$

Then, we can easily determine how the entanglement would evolve with time if we always drive the qubits with local operations so that at each time their state corresponds to the optimal one. We can simply solve the differential equation (5), obtaining $P(t) = \sin^2[h_{\max}t + \phi_0]$,

with $P(0) = \sin^2(\phi_0)$. Using the explicit dependence of E on P, we can directly then calculate $E(t)$. The evolution of the entanglement is fully characterized by $h_{\max}$, which is a quantity that only depends on the interaction Hamiltonian. That is, for a given H, $h_{\max}$ measures the capability of creating entanglement. In the following we give a simple way of determining $h_{\max}$, which allows us to classify the *entanglement capability* of any Hamiltonian. On the other hand, once the entanglement measure is specified, we can calculate the value P_0 of P for which we obtain the maximal rate by simply considering the function $f(P)$. For example, choosing the expression (3) for the entanglement, we find that P_0 solves the equation $\ln \frac{1-P_0}{P_0} = \frac{2}{1-2P_0}$, i.e. $P_0 \simeq 0.0832$ which gives $E(P_0) \simeq 0.413$. This shows that, in order to increase the entanglement of a two-qubit system in an optimal way, it is better to start with some initially entangled state rather than a product state [7]. Note that the optimal initial entanglement $E(P_0)$ is independent of H.

In the following we will show how to determine the entanglement capability $h_{\max}$ of a general Hamiltonian H acting on the qubits. First, we will show how, if we allow to supplement the evolution of H by local unitary operations, we can express H in a standard form that only depends on three parameters. Then we will derive an expression for $h_{\max}$ in terms of those parameters.

Except for a trivial constant, we can always express a general Hamiltonian as

$$H = \sum_{i=1}^{3} \alpha_i \sigma_i^A \otimes \mathbb{1}_B + \sum_{j=1}^{3} \beta_j \mathbb{1}_A \otimes \sigma_j^B + \sum_{i,j=1}^{3} \gamma_{i,j} \sigma_i^A \otimes \sigma_j^B. \quad (9)$$

Here, σ_i are the Pauli operators, and $\vec{\alpha}$, $\vec{\beta}$, and γ are two real vectors and a real matrix, respectively. We now show that by supplementing the evolution operator with local unitary operations we can obtain an effective Hamiltonian which has one of the two standard forms

$$\hat{H}^{\pm} = \mu_1 \sigma_1^A \otimes \sigma_1^B \pm \mu_2 \sigma_2^A \otimes \sigma_2^B + \mu_3 \sigma_3^A \otimes \sigma_3^B, \quad (10)$$

where $\mu_1 \geq \mu_2 \geq \mu_3 \geq 0$ are the (sorted) singular values of the matrix γ [8]. We first note that the terms corresponding to $\vec{\alpha}, \vec{\beta}$ in (9) give no contribution to $h_{\max}$ (8) and can therefore be neglected. Second, we apply the local operations U (V) and $U^\dagger$ ($V^\dagger$) to the first (second) qubit at the beginning and end of the evolution process, respectively. We select them such that $U^\dagger \sigma_i^A U = \sum_{k=1}^3 O_{k,i}^A \sigma_k^A$, $V^\dagger \sigma_j^B V = \sum_{l=1}^3 O_{j,l}^B \sigma_l^B$, where $O^{A,B}$ are orthogonal matrices of determinant one, each being plus or minus the orthogonal matrices in a singular value decomposition of γ. Thus the total (non-local) effect of the evolution for a time t is equivalent to the one obtained with the Hamiltonian $\hat{H}^+$ ($\hat{H}^-$) for the same time if $\det\gamma \geq 0$ ($\det\gamma < 0$). Without loss of generality, we may take H of the form $\hat{H}^+$ (10) in what follows [9].

Now let us determine $h_{\max}$ in terms of $\mu_{1,2,3}$. We can write $h(H, \varphi, \chi) = \sum_{k=1}^3 \mu_k \langle\varphi|\sigma_k^A|\varphi^\perp\rangle\langle\chi|\sigma_k^B|\chi^\perp\rangle$. Using the Cauchy–Schwarz inequality, it can be checked that the maximum of (the absolute value of) this function is reached for $|\chi\rangle = |\varphi^\perp\rangle$. In this case, using the fact that $|\varphi\rangle\langle\varphi| + |\varphi^\perp\rangle\langle\varphi^\perp| = \mathbb{1}$ we obtain $h(H, \varphi, \varphi) = \sum_{k=1}^3 \mu_k - \sum_{k=1}^3 \mu_k \langle\varphi|\sigma_k|\varphi\rangle^2$. Taking into account that $\mu_1 \geq \mu_2 \geq \mu_3$, we see that the maximum value occurs when $|\varphi\rangle = |0\rangle$ or $|\varphi\rangle = |1\rangle$, i.e. an eigenstate of σ_3. For that choice we obtain

$$h_{\max} = \mu_1 + \mu_2. \quad (11)$$

Summarizing, once we have transformed the Hamiltonian H to the standard form (10) we obtain that for a given value of E (and therefore of P),

$$|\Psi_E\rangle = \sqrt{P}|0,1\rangle + i\sqrt{1-P}|1,0\rangle, \quad (12a)$$

$$\Gamma_E = f(P) h_{\max} \quad (12b)$$

where $h_{\max} = \mu_1 + \mu_2$. The maximum rate $\Gamma_{\max}$ is obtained for $P = P_0$, where P_0 is the value that maximizes $f(P)$. Thus, $|\Psi_{\max}\rangle$ and $\Gamma_{\max}$ are given by (12) with $P = P_0$. For example, for the entanglement measure (3), $P_0 \simeq 0.0832$ which leads to $f(P_0) \simeq 1.9123$.

So far, we have calculated the most efficient way of entangling two qubits if we can use local unitary operations acting on each of the qubits. We have not allowed, however, local operations which entangle each of the qubits with local ancillas. We will now show that this possibility permits us to increase the maximum entanglement rate $\Gamma_{\max}$ for certain kind of Hamiltonians. We will first generalize the formulas derived above to the case of multilevel systems, given that the system qubit–plus–ancilla is of this sort. We consider a state $|\Psi\rangle$ with Schmidt decomposition $|\Psi\rangle = \sum_{n=1}^N \sqrt{\lambda_n}|\varphi_n, \chi_n\rangle$. As before, any entanglement measure E will only depend on the Schmidt coefficients $\lambda_n \geq 0$. In particular, in the following we will use the entropy of entanglement, $E(\vec{\lambda}) = -\sum_{n=1}^N \lambda_n \log_2(\lambda_n)$. Using the definition (1) of entanglement rate, we have

$$\tilde{\Gamma} = \sum_{n=1}^{N} \frac{\partial E}{\partial \lambda_n} \frac{d\lambda_n}{dt} = \frac{1}{N} \sum_{n,m=1}^{N} \left[\frac{\partial E}{\partial \lambda_n} - \frac{\partial E}{\partial \lambda_m}\right] \frac{d\lambda_n}{dt}, \quad (13)$$

where we have used the fact that the sum of all the Schmidt coefficients is constant. Using perturbation theory as before, we find $\frac{d\lambda_n}{dt} = 2\sqrt{\lambda_n}\mathrm{Im}[\langle\varphi_n, \chi_n|H|\Psi\rangle] = 2\sum_{m=1}^N \sqrt{\lambda_n \lambda_m}\mathrm{Im}[\langle\varphi_n, \chi_n|H|\varphi_m, \chi_m\rangle]$.

Rather than proceeding in complete generality we now consider an example which demonstrates that adding ancillas may allow one to increase entanglement more efficiently than is possible without the use of ancillas. We will consider the case in which the ancillas are also qubits. We write $P = \lambda_1$ and concentrate on the case in which $\lambda_2 = \lambda_3 = \lambda_4 = (1-P)/3$. In that case, Eq. (13) simplifies to

$$\tilde{\Gamma} = \tilde{f}(P)\tilde{h}(H, \varphi_n, \chi_n) \tag{14}$$

where now

$$\tilde{f}(P) = 2\sqrt{P(1-P)/3}\log_2[(1-P)/(3P)], \tag{15a}$$

$$\tilde{h}(H, \varphi_n, \chi_n) = \sum_{n=2}^{4} \mathrm{Im}[\langle\varphi_1, \chi_1|H|\varphi_n, \chi_n\rangle]. \tag{15b}$$

We can always choose the phase of the states $|\varphi_n\rangle$ such that all the terms on the sum add with the same sign. We can therefore replace the imaginary parts of the terms in the above expression by their absolute values, and in (14) we can replace $\tilde{f}(P)$ by $|\tilde{f}(P)|$. We find that $\tilde{P}_0 \simeq 0.8515$ (which corresponds to an entropy of entanglement $E(\tilde{P}_0) \simeq 0.8415$) maximizes $|\tilde{f}(P)|$ (15a) and leads to $|\tilde{f}(\tilde{P}_0)| \simeq 1.6853$. Proceeding as before, we can easily maximize $\tilde{h}$. We obtain that the maximum value is $\tilde{h}_{\max} = \mu_1 + \mu_2 + \mu_3$, which occurs when $|\varphi_n\rangle = |\chi_n\rangle$ are orthogonal maximally entangled states between the qubit and the ancilla. For example in the case that $\det\gamma \geq 0$, the choice $|\varphi_1\rangle = |\phi^+\rangle, |\varphi_2\rangle = i^{\frac{3}{2}}|\psi^+\rangle, |\varphi_3\rangle = i^{\frac{1}{2}}|\psi^-\rangle, |\varphi_4\rangle = i^{\frac{3}{2}}|\phi^-\rangle$, where $\{|\phi^\pm\rangle, |\psi^\pm\rangle\}$ are Bell states [10], together with $P = P_0 = 0.8515$ leads to a maximal [under the previous assumptions on the λ_i's] entanglement rate $\tilde{\Gamma} = \tilde{\Gamma}_{\max}$.

Let us compare the cases in which we use ancillas and the one in which we do not use them. On the one hand, we have that $|\tilde{f}(\tilde{P}_0)| < |f(P_0)|$. But on the other, $\tilde{h}_{\max} \geq h_{\max}$. Thus, if $\mu_3 \neq 0$ it may be the case that the use of ancillas can help to increase the maximum rate of entanglement $\Gamma_{\max}$ as well as the rate Γ_E for a given entanglement E of state $|\Psi\rangle$. This is in fact the case, if we have, for example, $\mu_1 = \mu_2 = \mu_3$ (i.e. $\tilde{h}_{\max} = 3/2h_{\max}$). In this case we obtain $\tilde{\Gamma}_{\max} \simeq 1.3220\Gamma_{\max}$. In a similar way, one can check for this specific Hamiltonian that $\tilde{\Gamma}_E \geq \Gamma_E$ if the initial entanglement satisfies $E \geq 0.08$.

Finally, it is easy to show that all entangling Hamiltonians are *qualitatively* equivalent when assisted by local operations. In particular, given two Hamiltonians H and H' with either $h_{\max} = \alpha h'_{\max}$ or $\tilde{h}_{\max} = \alpha\tilde{h}'_{\max}$, one can simulate the action of H' for any time t by applying H for at most $3\alpha^{-1}t$. This can be seen as follows: Applying H of the form (10) for $\delta t/2$ followed by a local unitary operation σ_1 in A before and after another application of H for $\delta t/2$ is equivalent to the application of the Hamiltonian $H_1 = \mu_1\sigma_1 \otimes \sigma_1$ for the time δt, provided that δt is infinitessimally small. Since H_1 is locally equivalent to $H_k = \mu_1\sigma_k \otimes \sigma_k$, applying sequentially H_k for $\delta t\mu'_k/\mu_1$ is equivalent to the application of H' for the time δt. Using the restrictions on μ_k, μ'_k the claim readily follows. The question of efficient simulation of another Hamiltonian in the same time t will be addressed in future work.

In summary, we have found the optimal way of using any non–local interaction to entangle a pair of qubits. The idea is to use local operators to drive the instantaneous state to the one that maximizes the entanglement rate, at each moment of the evolution. We have found that the entanglement capacity of any given Hamiltonian is determined by the sum of the two largest singular values of the matrix γ defined in (9). Finally, we have shown that for certain Hamiltonians one can overcome this maximal entanglement rate by using ancillas prepared in maximally entangled states with the qubits.

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Note added: The idea of universal simulation of Hamiltonians as discussed in this manuscript originated in a larger collaboration, including several other people and is developed in detail in [11]. After completion of this manuscript we have also learned that the notion of universal simulation of Hamiltonians has been independetly addressed by Dodd et al. [12].

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[3] Recently, related questions concerning the average entangling power of unitary evolutions were also considered in P. Zanardi, C. Zalka and L. Faoro, quant-ph/0005031

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[5] Note that the definitions of Γ and $\Gamma_{\max}$ are not restricted to qubits but are also valid for d–level systems, $d \geq 2$.

[6] C. H. Bennett, H. Bernstein, S. Popescu and B. Schumacher, Phys. Rev. A **53**, 2046-2052 (1996). We note that classical communication (CC) is needed for dilution; however it was shown (H-K. Lo and S. Popescu, Phys.Rev.Lett **83**, 1459, (1999)) that in the asymptotic limit no CC is needed; thus CC is unlikely to enhance the asymptotic entanglement capability.

[7] The idea of enhancing the entanglement rate by starting with an already entangled state as well as the qualitative equivalence of Hamiltonians was first discussed by S. Braunstein, I. Chuang, D. DiVincenzo, D. Gottesman, B. Terhal and C. H. Bennett during the ISI meeting in Turin (1998).

[8] Given a real matrix γ one can always find two orthogonal matrices O^1 and O^2 such that $\gamma = O^1\mu O^2$, where μ is a diagonal matrix with positive entries; this is called a singular value decomposition of γ and the diagonal elements of μ are the singular values.

[9] Due to the fact that $\hat{H}^- = \hat{H}^{+^{T_A}}$, where T_A denotes the partial transposition with respect to system A, we have

that $\langle\varphi,\chi|\hat{H}^{+}|\varphi^{\perp},\chi^{\perp}\rangle = \langle\varphi^{\perp *},\chi|\hat{H}^{-}|\varphi^{*},\chi^{\perp}\rangle$. That is, if $h_{\max}$ for $\hat{H}^{+}$ is obtained by states $|\varphi,\chi\rangle$, $h_{\max}$ for $\hat{H}^{-}$ is obtained by the states $|\varphi^{\perp *},\chi\rangle$. Thus, we can restrict our analysis to $\hat{H}^{+}$.

[10] The Bell states are defined as $|\phi^{\pm}\rangle = 1/\sqrt{2}(|00\rangle \pm |11\rangle)$ and $|\psi^{\pm}\rangle = 1/\sqrt{2}(|01\rangle \pm |10\rangle)$.

[11] C. H. Bennett et al., quant-ph/0107035.

[12] J. L. Dodd et al., quant-ph/0106064.